

# On Commutative Monoid Congruences<sup>1</sup> of Semigroups

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## Abstract

A subset  $A$  of a semigroup  $S$  is called a medial subset of  $S$  if  $xy \in A$  implies  $xy \in A$  for every  $a, b, x, y \in S$ . By the separator of a subset  $A$  of a semigroup  $S$ , we mean the set of all elements  $x$  of  $S$  which satisfy the conditions  $xA \subseteq A$ ,  $Ax \subseteq A$ ,  $x\bar{A} \subseteq \bar{A}$ ,  $\bar{A}x \subseteq \bar{A}$ , where  $\bar{A}$  denotes the complement of  $A$  in  $S$ . In this paper we show that if  $\{A_i, i \in I\}$  is a family of medial subsets of a semigroup  $S$  such that  $A = \bigcap_{i \in I} \text{Sep}(A_i)$  is not empty then  $P_{\{A_i, i \in I\}}$  defined by  $(a, b) \in P_{\{A_i, i \in I\}}$  ( $a, b \in S$ ) if and only if, for every  $i \in I$  and  $x, y \in S$ ,  $xy \in A_i \Leftrightarrow xby \in A_i$  is a commutative monoid congruence of  $S$  such that  $A$  is the identity element of  $S/P_{\{A_i, i \in I\}}$ . Conversely, every commutative monoid congruence of a semigroup can be so constructed. We also show that if  $S$  is a permutative semigroup then the monoid congruences of  $S$  are exactly the congruences  $P_{\{A_i, i \in I\}}$  defined for arbitrary family  $\{A_i, i \in I\}$  of arbitrary subsets of  $S$  satisfying  $\bigcap_{i \in I} \text{Sep}(A_i) \neq \emptyset$ .

By the idealizer  $Id(A)$  of a subset  $A$  of a semigroup  $S$  we mean the set of all elements  $x$  of  $S$  which satisfy the conditions  $xA \subseteq A$ ,  $Ax \subseteq A$ . Denoting the complement of  $A$  in  $S$  by  $\bar{A}$ , the subset  $\text{Sep}(A) = Id(A) \cap Id(\bar{A})$  of  $S$  is called the separator of  $A$  ([2]). In other words, the separator of  $A$  is the set of all elements  $x$  of  $S$  which satisfy the conditions  $xA \subseteq A$ ,  $Ax \subseteq A$ ,  $x\bar{A} \subseteq \bar{A}$ ,  $\bar{A}x \subseteq \bar{A}$ .

**Lemma 1** ([2]) *For any subset  $A$  of a semigroup  $S$ ,  $\text{Sep}(A)$  is either empty or a subsemigroup of  $S$ .*  $\square$

**Lemma 2** ([2]) *If  $A$  is a subset of a semigroup such that  $\text{Sep}(A) \neq \emptyset$  then either  $\text{Sep}(A) \subseteq A$  or  $\text{Sep}(A) \subseteq \bar{A}$ .*  $\square$

A subset  $U$  of a semigroup  $S$  is said to be a left (right) unitary subset of  $S$  if  $a, ab \in U$  ( $a, ba \in U$ ) implies  $b \in U$  for every  $a, b \in S$ . The subset  $U$  is called a unitary subset of  $S$  if it is both left and right unitary in  $S$ .

**Lemma 3** ([2]) *A subsemigroup  $A$  of a semigroup  $S$  is unitary in  $S$  if and only if  $A = \text{Sep}(A)$ .*  $\square$

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Let  $\{A_i, i \in I\}$  be a family of non-empty subsets of a semigroup  $S$ . It is easy to see that the relation  $P_{\{A_i, i \in I\}}$  on  $S$  defined by  $(a, b) \in P_{\{A_i, i \in I\}}$  ( $a, b \in S$ ) if and only if, for every  $i \in I$  and  $x, y \in S$ ,  $xy \in A_i \Leftrightarrow xby \in A_i$  is a congruence of  $S$ .

**Definition 1** A subset  $A$  of a semigroup  $S$  will be called a medial subset of  $S$  if  $xaby \in A$  if and only if  $xbay \in A$  for every  $a, b, x, y \in S$ .

**Theorem 1** Let  $\{A_i, i \in I\}$  be a family of medial subsets of a semigroup  $S$  such that  $A = \cap_{i \in I} \text{Sep}(A_i)$  is not empty. Then  $P_{\{A_i, i \in I\}}$  is a commutative monoid congruence of  $S$  such that  $A$  is the identity element of  $S/P_{\{A_i, i \in I\}}$ . Conversely, every commutative monoid congruence of a semigroup can be so constructed.

**Proof.** Let  $\{A_i, i \in I\}$  be a family of medial subsets of a semigroup  $S$  such that  $A = \cap_{i \in I} \text{Sep}(A_i)$  is not empty. As  $xaby \in A_i$  iff  $xbay \in A_i$  for every  $a, b, x, y \in S$  and  $i \in I$ , we get that  $P_{\{A_i, i \in I\}}$  is a commutative congruence on  $S$ . Let  $a$  and  $b$  be arbitrary elements of  $S$  such that  $a \in A_i$  and  $b \notin A_i$  for some  $i \in I$ . Then, for every  $g, h \in A$ , we have  $gah \in A_i$  and  $gbh \notin A_i$  and so  $(a, b) \notin P_{\{A_i, i \in I\}}$ . Thus  $A_i$  is a union of  $P_{\{A_i, i \in I\}}$ -classes for every  $i \in I$ . Let  $a, b \in A$  be arbitrary elements. Assume  $xy \in A_i$  for some  $i \in I$  and  $x, y \in S$ . Since  $b \in \text{Sep}(A_i)$ , we get  $xayb \in A_i$ . As  $A_i$  is a medial subset of  $S$ , we get  $xyab \in A_i$  and so  $xy \in A_i$ , because  $ab \in \text{Sep}(A_i)$ . Then  $xyba \in A_i$ ,  $xya \in A_i$  and  $xyb \in A_i$ , because  $ba \in \text{Sep}(A_i)$ ,  $A_i$  is medial and  $a \in \text{Sep}(A_i)$ . Thus  $(a, b) \in P_{\{A_i, i \in I\}}$ . Let  $a \in A$  and  $b \notin A$  be arbitrary elements. Then there is an index  $j \in I$  such that  $b \notin \text{Sep}(A_j)$ . We have four cases:  $bA_j \not\subseteq A_j$ ,  $A_jb \not\subseteq A_j$ ,  $b\overline{A_j} \not\subseteq \overline{A_j}$ ,  $\overline{A_j}b \not\subseteq \overline{A_j}$ . In case  $bA_j \not\subseteq A_j$ , there is an element  $c \in A_j$  such that  $bc \notin A_j$  and so  $abc \notin A_j$ . As  $aac \in A_j$ , we get  $(a, b) \notin P_{\{A_i, i \in I\}}$ . We get the same result in the other three cases. Thus  $A$  is a  $P_{\{A_i, i \in I\}}$ -class. Let  $a \in A$  and  $s \in S$  be arbitrary elements. Then, for every  $x, y \in S$ ,  $xsay \in A_i$  iff  $xsaya \in A_i$  iff  $xsyaa \in A_i$  iff  $xsy \in A_i$ . Thus  $(sa, s) \in P_{\{A_i, i \in I\}}$ . We can prove, in a similar way, that  $(as, s) \in P_{\{A_i, i \in I\}}$ . Hence  $A$  is the identity element in the factor semigroup  $S/P_{\{A_i, i \in I\}}$ . Hence  $P_{\{A_i, i \in I\}}$  is a commutative monoid congruence of  $S$ .

Conversely, let  $\sigma$  be a commutative monoid congruence of a semigroup  $S$ . Let  $A$  denote the  $\sigma$ -class which is the identity element of  $S/\sigma$ . Let  $\{A_i, i \in I\}$  denote the family of all  $\sigma$ -classes of  $S$ . It is obvious that  $A_i, i \in I$  are medial subsets of  $S$ . Let  $a \in A$  be an arbitrary element. As  $aA_i \subseteq A_i$  and  $A_ia \subseteq A_i$  for every  $i \in I$ , we get  $a \in \cap_{i \in I} \text{Sep}(A_i)$ . Hence  $A \subseteq \cap_{i \in I} \text{Sep}(A_i)$ . Assume that there is an element  $b$  of  $S$  such that  $b \in \cap_{i \in I} \text{Sep}(A_i)$  and  $b \notin A$ . Then there is an index  $j \in I$  such that  $b \in A_j \neq A$  and so  $A_j \cap \text{Sep}(A_j) \neq \emptyset$ . Then, by Lemma 2,  $\text{Sep}(A_j) \subseteq A_j$  which implies  $A \subseteq A_j$  which is impossible. Hence  $A = \cap_{i \in I} \text{Sep}(A_i)$ . In the first part of the proof, it was proved that  $A_i, i \in I$  are unions of  $P_{\{A_i, i \in I\}}$ -classes. Hence  $P_{\{A_i, i \in I\}} \subseteq \sigma$ . As every  $A_i, i \in I$  is a  $\sigma$ -class, it is obvious that  $\sigma \subseteq P_{\{A_i, i \in I\}}$ . Consequently  $\sigma = P_{\{A_i, i \in I\}}$ .  $\square$

A subset  $A$  of a semigroup  $S$  is said to be a reflexive subset of  $S$  if  $ab \in A$  implies  $ba \in A$  for every  $a, b \in S$ .

**Corollary 1** *For any medial subset  $A$  of a semigroup  $S$ ,  $Sep(A)$  is either empty or a reflexive unitary subsemigroup of  $S$ .*

**Proof.** Let  $A$  be a medial subset of a semigroup  $S$  such that  $Sep(A) \neq \emptyset$ . By Theorem 1,  $P_A$  is a monoid congruence of  $S$  such that  $Sep(A)$  is the identity element of the factor semigroup  $S/P_A$ . Then  $Sep(Sep(A)) = Sep(A)$  and, by Lemma 3,  $Sep(A)$  is a unitary subsemigroup of  $S$ . As  $P_A$  is a commutative congruence by Theorem 1,  $Sep(A)$  is reflexive.  $\square$

**Definition 2** *A semigroup  $S$  is called a permutative semigroup ([3]) if it satisfies a non-trivial permutation identity, that is, there is a positive integer  $n \geq 2$  and a non-identity permutation  $\sigma$  of  $\{1, 2, \dots, n\}$  such that  $S$  satisfies the identity  $x_1 x_2 \dots x_n = x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(n)}$ .*

It is obvious that every permutative monoid is commutative. Next, we construct the monoid congruences of permutative semigroups.

**Lemma 4** [4] *Let  $S$  be a permutative semigroup. Then there exists a positive integer  $k$  such that, for every  $u, v \in S^k$  and  $x, y \in S$ , we have  $uxyv = uyxv$ .*

**Theorem 2** *Let  $\{A_i, i \in I\}$  be a family of subsets of a permutative semigroup  $S$  such that  $A = \bigcap_{i \in I} Sep(A_i)$  is not empty. Then  $P_{\{A_i, i \in I\}}$  is a monoid congruence of  $S$  such that  $A$  is the identity element of  $S/P_{\{A_i, i \in I\}}$ . Conversely, every monoid congruence of a permutative semigroup can be so constructed.*

**Proof.** Let  $S$  be a permutative semigroup. Then, by Lemma 4, there is a positive integer  $k$  such that, for every  $u, v \in S^k$  and  $x, y \in S$ , we have  $uxyv = uyxv$ . Let  $X$  be a non-empty subset of  $S$  such that  $Sep(X) \neq \emptyset$ . Assume  $uxyv \in X$  for some  $u, v, x, y \in S$ . Then, for some  $t \in Sep(X)$ , we have  $(t^{k-1}u)yx(vt^{k-1}) = (t^{k-1}u)xy(vt^{k-1}) \in X$  which implies  $uyxv \in X$ . Hence  $X$  is a medial subset of  $S$ . Assume that  $\{A_i, i \in I\}$  is a family of subsets of  $S$  such that  $A = \bigcap_{i \in I} Sep(A_i)$  is not empty. Then, by the above, every  $A_i$  is a medial subset of  $S$  and so, by Theorem 1,  $P_{\{A_i, i \in I\}}$  is a (commutative) monoid congruence of  $S$  such that  $A$  is the identity element of  $S/P_{\{A_i, i \in I\}}$ . The converse follows from Theorem 1.  $\square$

**Corollary 2** *For any subset  $A$  of a permutative semigroup  $S$ ,  $Sep(A)$  is either empty or a reflexive unitary subsemigroup of  $S$ .*

**Proof.** As a subset  $A$  of a permutative semigroup  $S$  with  $Sep(A) \neq \emptyset$  is medial, the assertion follows from Corollary 1.  $\square$

## References

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